WORLD INCOME COMPONENTS:
MEASURING AND EXPLOITING RISK-SHARING OPPORTUNITIES

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World Income Components: Measuring and Exploiting Risk-Sharing Opportunities

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A method is constructed for decomposing the variance of changes in incomes in the world into components, to indicate the most important risk-sharing opportunities among people of the world. A constant absolute risk premium (CARP) model, an intertemporal general-equilibrium model of the world, is presented to permit optimal contract design. For a contract designer maximizing a social welfare function, the optimal contracts maximize the equilibrium world real interest rate. Securities are defined in terms of eigenvectors of a transformed variance matrix. The method is applied using Penn World Table data on the G-7 countries, 1950–1992. (*JEL* F00, G00, G10)

Because most people's incomes originate primarily from untradable sources such as labor and difficult-to-trade sources such as real estate, and because over long intervals of time the real value of each individual's income is substantially uncertain, the dominant concern in the design of risk-management contracts ought to be allowing people to share these income risks as much as possible. To make such risk sharing as effective as possible, it is logical to define risk-management contracts in the form of securities based on these incomes themselves; these securities could be defined by contracts that represent long-term or perpetual claims on incomes or income aggregates.[1] Such securities can be designed to correlate better than existing assets with major risks and thus serve better as hedging vehicles.

We cannot, however, have a market for contracts on each individual's income. There would be far too many markets: the markets will have to be markets for contracts on aggregates (or indexes) of individual incomes. The question then arises, and is the subject of this paper, how shall we decide which aggregates to trade on risk markets?

With billions of people in the world today, there are a myriad of ways to define aggregates of incomes that could be traded on our financial markets. Our national income statisticians have already chosen some aggregates, the simple sums of incomes of people within nations, but they did so based purely on political boundaries and without any concerns for risk management. How do we know which aggregates would be most important to create markets for? Is it possible to be systematic in our means of defining new markets, seeking out the new markets that maximize world welfare without regard for traditional definitions of contracts, letting an estimated economic model define them?

We develop a constant absolute risk premium (CARP) model, an intertemporal general-equilibrium model, that is designed to allow us to see how we might answer these questions and provides a framework for rigorous econometric research on market design. The model represents a large number of people as trading in riskless one-period bonds and in a small number of risky (risk-management) contracts. It is a risky endowment economy, where each individual has an exogenously given random income, at first with no risk management beyond

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[1] See Shiller (1993a) for a discussion of how such contracts could be implemented.

borrowing and lending possible. Then new contracts representing claims on linear income combinations (CLICs) will be injected. The prices of the CLICs, their equilibrium expected returns, the real interest rate, and the welfare gain to creating markets are all derived. The model then yields a method to define the welfare-optimizing CLICs, the world income components (WICs), to be used in the definition of new securities. As an example, we apply the model to the readily available data on world incomes of advanced countries, data on national incomes of the G-7 countries, as measured by the real gross domestic product data of Robert Summers and Alan Heston (1950–1992), based on a simple model of the relation of individual incomes to national incomes. In the future, further econometric work combining the fragmentary data we have on individuals around the world in conjunction with this model might be used to define better aggregates of individual incomes on which to base the definitions of new securities.

Our method of identifying risk-sharing arrangements [based on Athanasoulis (1995) and Shiller and Athanasoulis (1995)] is related to principal components analysis [see also Gabrielle Demange and Guy Laroque (1995)]. Some of the securities that our method produces can be described as insurance policies for certain groups of people; calling a security an insurance policy is most appropriate when the variation in the component is highly negatively correlated with the income of one group of people, and those people buy the security to reduce their income risk. Some of the securities can also be described as swaps of income of certain groups of people for income of other groups; calling one of our securities a swap is most appropriate when the component gives negative weights to roughly half of the incomes. However, our analysis does not start from any preconceived notions whether the securities will look like insurance policies or like swaps; our method will derive the optimal form of the contracts from the variance matrix of incomes.

Our study of risk-sharing opportunities among individual incomes is potentially very important. There is very little effective risk sharing of individual income risks. There is not even much diversification across nations today [see, e.g., Maurice Obstfeld (1994a, b) and Linda Tesar and Ingrid Werner (1995)]. It is obvious that national governments do not make significant risk-sharing arrangements with each other; even within the European Union, Xavier Sala-i-Martin and Jeffrey Sachs (1992) estimate, a one-dollar adverse shock to the national income of one country creates, all things considered, less than a 0.005-dollar reduction of that country's tax payment to the European Community. Although the family or village units may share risks within a country, there are still important unexploited risk-sharing opportunities within countries [see for example Robert M. Townsend (1994)].

Because there do not now exist any markets for claims on any large income aggregates, and because there is very little income risk sharing, when we set up any such contracts we must consider how they would work pretty much in isolation. Existing markets for stocks, bonds, and readily marketable real estate are markets for claims on the rents of factors of production other than labor or are residual claims, minor components of national incomes, and there is no reason to expect dividends in these markets to correlate well with labor income. There is instead some evidence that they do not correlate well [see Shiller (1993b) and L. Bottazzi et al. (1996)].

In attempting to define a small number of WIC securities, we are essentially seeking to define the best first world risk-sharing market to set up, as well as the best second and/or third markets. We confine our attention to only one or a few contracts, in that it is useful for us to be able to prescribe in simple terms the most important risk-management actions that should be taken by large groups of people. Simple prescriptions are what most people take from existing models. The capital asset pricing model (CAPM) in finance, to which our method is related, is most often used by practitioners not to arrive at complicated definitions of optimal portfolios, but just for the simple prescription that investors should hold the market portfolio of investable assets, and we now have many indexed funds that are designed to allow them to do just this. The problem with this commonly given prescription is that it is not really the logical consequence of the foundations of the CAPM, because it disregards the correlation of investment returns with innovations in other income, other income that is much larger in the aggregate than income from existing investable assets. We seek here to devise a method to replace this simple prescription associated with the CAPM with a more sensible simple prescription, though any such prescription cannot be taken until the new securities are created.

We derive in Section II below an expression for the risk premia of the CLICs in equilibrium. The CLICs are then chosen so as to maximize social welfare (Section III); this then defines our WIC securities. It turns out that the WICs are defined in terms of eigenvectors of a variance matrix of deviations of individual incomes from per capita world income. We also show that with these securities, the world interest rate achieves a maximum, the risk-optimal interest rate. With constant absolute risk aversion (CARA) utility, if the variance matrix of incomes does not change through time, then, even if there are major shifts in endowments across groups, the WICs will not change through time, nor will the individuals' optimal investments change. The markets for WIC securities need to be set up only once and people have to decide only once how much to invest in each security. Thus, the creation of the new WIC securities is a relatively simple, and potentially very important, step to achieve welfare gains. Having made a specification of utility functions, we are able to derive estimates of the welfare increase in dollars generated by the creation of the new contracts.

In Section IV we discuss how to apply our method of defining the WIC securities to the data. We present a model of individual income, the three-level income model, that allows estimation of a restricted variance matrix of individual incomes for the G-7 countries. A discussion follows to interpret these results as suggesting opportunities for potentially important new contracts. In Section V we conclude.

I. Definition of Contracts and Risk Structure

There are two kinds of securities that are traded in the economy: CLICs, which are long-lived securities, and riskless one-period bonds. Both kinds of securities are in zero net supply: for every long there is a short. All CLICs are assumed to be traded (constructed) for the first time at time 0, and people immediately make optimal use of these contracts from that date onward. The construction of the CLIC securities at time zero is unanticipated by all individuals in the economy. In contrast, bonds are assumed already traded: the bond market is the one pre-existing market that we represent in our model. Some methods that take account of other pre-existing securities are outlined in Athanasoulis and Shiller (2000). There are N kinds of perpetual claims, and I individuals, where presumably I is much larger than N. Individuals are infinite lived, and in each period t, from 0 onward, reevaluate their holdings of these securities in view of the prices of the perpetual claims P_{tn}, $n = 1, \ldots, N$, and the interest rate r_t from t to $t + 1$.

The model takes as exogenous the income process $\{y_{ti}\}_{t_0}^{\infty}$ for each individual i, $i = 1, \ldots, I$. The date that bonds began to be traded is $t_0 < 0$. Income y_{ti} is derived from sources other than the risk-management contracts or bonds, let us say labor; it may also be called the endowment of individual i at time t. Let $\mathbf{y}_t$ be the I-element column vector whose ith element is y_{ti}. We will assume that $\mathbf{y}_t$ is a Gaussian random walk:

$$\mathbf{y}_t = \mathbf{y}_{t-1} + \boldsymbol{\varepsilon}_t, \tag{1}$$

where $\boldsymbol{\varepsilon}_t$ is independently and identically distributed (i.i.d.) normal, with zero mean and with variance matrix $\boldsymbol{\Sigma}$ that is constant through time. In the rest of the paper we use the convention that vectors are either I-element column vectors, N-element row vectors, and, when appropriate, matrices are $I \times N$.

Each CLIC specifies both a riskless payment and a risky payment that must be made from the short in the CLIC to the long each time period starting with time $t = 1$. We will call the sum of the riskless and the risky payment the dividend of the nth CLIC, D_{tn}. We call the riskless payment the risk premium $\bar{D}_n$, paid on the nth CLIC in each period $t \geq 1$. We will choose $\bar{D}_n$ below so that the CLIC has a zero price at time 0, $P_{0n} = 0$. $\bar{D}_n$ may thus be thought of as a regular insurance premium paid by the shorts in the CLIC or regular risk premium received by the longs in the CLIC; no other payment or compensation for risk bearing is expected at time 0. The assumption that the risk premium is constant through time is natural, given our assumption that the variance matrix Σ is constant through time.

We will denote the risky payment made from the short to the long in each period from time $t = 1$ onward by X_{tn}, which may be positive or negative. X_{tn} is defined in terms of individual endowments in such a way that $E_0(X_{tn}) = 0$ and $\text{Var}_{t-1}(X_{tn}) = 1$. Setting this variance to 1 is just a normalization rule to define the size of one CLIC. To define the aggregates we assume that an I-element column vector $\mathbf{A}_n$ is defined in the nth CLIC at time 0, and the CLIC specifies that $X_{tn} = (\mathbf{y}_t - \mathbf{y}_0)'\mathbf{A}_n$. The risky payment is defined as a linear combination of the (unexpected) changes in incomes, given that risk management began at time 0. When $\mathbf{A}_n$ is chosen optimally, as will be defined below, the risky payment X_{tn} will be our nth world income component (WIC). Our objective below will be to define this random payment stream optimally by choosing the best vector $\mathbf{A}_n$. We adopt the convention that $\mathbf{A}_n$ is defined so that the risk premium $\bar{D}_n$ is nonnegative. If $\bar{D}_n$ were negative we would multiply $\mathbf{A}_n$ by minus one. This is just a convention defining who is called long and who short. We may write the processes for the dividend, the risky, and the riskless payment as

$$\text{(2)} \quad \bar{D}_{tn} = \bar{D}_n \qquad \text{for } t \geq 1,$$

$$\text{(3)} \quad X_{tn} = (\mathbf{y}_t - \mathbf{y}_0)'\mathbf{A}_n \qquad \text{for } t \geq 1,$$

$$\text{(4)} \quad D_{tn} = \bar{D}_n + X_{tn} \qquad \text{for } t \geq 1.$$

Commitments to make payments specified in the CLICs last forever, and at each time period t one can avoid making future payments only by selling the CLIC that one is long in or buying a CLIC that one is short in; there is no free disposal and no default. Thus, the price P_{tn} of the nth CLIC at time t can be either positive or negative after time 0, depending on which way incomes turn. If price becomes positive after time 0 it means that the CLIC is valuable to longs because it is expected to make positive risky payments to them. If price becomes negative after time 0 it means that the CLIC is valuable to shorts because it is expected to make negative risky payments to the longs. Knowing this, people will at time 0 try to take positions in the CLICs that help them to offset their endowment risks, and each period thereafter will reevaluate the ability of the CLICs to do this; prices of the CLICs after time 0 and the interest rate will be determined by the market-clearing condition.

Our timing convention is as follows: In each period t individual i's endowment y_{ti} is realized. The individual also receives payments from his or her securities holdings, that is, receiving both dividends and interest from the CLICs and the one-period bond. Individual i then decides how much to consume and how many of the CLICs and the riskless bonds to buy or sell, taking account of expected future payoffs of the securities as well as the prices P_{tn}, $n = 1, \ldots, N$, and interest rate r_t that are simultaneously determined in the competitive market. At time 0, in contrast, no dividends are received, and individual i chooses quantities of the CLICs and the bond to buy or sell. The risk premium $\bar{D}_n$ is chosen by the contract designer so that equilibrium prices of the CLICs at time zero P_{0n}, $n = 1, \ldots, N$, are zero. Thus, at time zero the very important risk-sharing commitments are made and the only resources that change hands are those for the purchase and sale of bonds as well as payment of interest and principal for bonds issued at time $t - 1$.

Let $\mathbf{X}_t$ denote the N-element row vector whose nth element is X_{tn}. Then, $\mathbf{X}_t = (\mathbf{y}_t - E_0\mathbf{y}_t)'\mathbf{A}$, where $\mathbf{A}$ is the $I \times N$ matrix whose nth column is $\mathbf{A}_n$. Let $\mathbf{P}_t$ denote the N-element row vector whose nth element is P_{tn}; $\bar{\mathbf{D}}$ denotes the N-element row vector whose nth element is $\bar{D}_n$; $\mathbf{D}_t$ denotes the N-element row vector whose nth element is D_{tn}. Note that $\mathbf{D}_t = \bar{\mathbf{D}} + \mathbf{X}_t$. We also adopt the normalization that $\mathbf{A}'\Sigma\mathbf{A} = \mathbf{I}$, where $\mathbf{I}$ is the $N \times N$ identity matrix. This is just a normalization that does not restrict our contracts.

II. The Constant Absolute Risk Premium (CARP) Model

In this section we develop a general-equilibrium model of the world that has the property that the risk premia are constant through time in absolute (dollar) terms. This property yields important simplifications for our purposes because it makes possible a closed-form solution with a constant investment opportunity set in dollars rather than in percent, the latter used in Robert C. Merton (1971) and others. Thus, we will see that there are simple relations such as $P_{tn} = X_{tn}/r$, where r is a constant riskless interest rate.

We will assume that each individual i at time t maximizes

$$(5)\qquad U_{ti} = \mathrm{E}_t\left[\sum_{\tau=0}^{\infty} u_t(c_{t+\tau i})/(1+\rho)^{\tau}\right],$$

where $u_t(c_{t+\tau i})$ is felicity, or instantaneous utility, of individual i at time t of consumption at time $t + \tau$, and ρ is the discount rate (i.e., the subjective rate of time preference). We assume that the felicity $u_t(c_{t+\tau i})$ is defined by a negative exponential (CARA) function:

$$(6)\qquad u_t(c_{t+\tau i}) = -\exp(-\gamma c_{t+\tau i}), \qquad \gamma > 0.$$

This felicity function will allow us to compute an explicit closed-form general equilibrium solution.[2] We assume that $\ln(1 + \rho) > \frac{1}{2}\gamma^2(1/I)\sum_{i=1}^{I} \sigma_{\varepsilon_i}^2$, where $\sigma_{\varepsilon_i}^2$ is the variance of the innovation to income for agent i. As will be seen below, this ensures a positive market-clearing real interest rate. We also assume the interest rate process is bounded.

Let us define $\mathbf{q}_{ti}$ as the N-element row vector, whose nth element is the number of securities n owned by individual i at time t. Let us also define bond demand (in units of the consumption good) of individual i at time t, earning interest from time t to time $t + 1$ as B_{ti}. The budget constraint of individual i at time t is

$$(7)\qquad c_{ti} = y_{ti} + (1 + r_{t-1})B_{t-1i} - B_{ti} + \mathbf{q}_{t-1i}(\mathbf{P}_t + \mathbf{D}_t)' - \mathbf{q}_{ti}\mathbf{P}_t',$$

which holds for all t, though we require that $\mathbf{q}_{ti}$ is zero for $t < 0$.[3] Individual i maximizes expected lifetime utility (5), subject to the budget constraint (7), initial conditions B_{t-1i} and y_{ti} are given, and the no-Ponzi-game condition, which is

$$(8)\qquad \lim_{\tau\to\infty} \mathrm{E}_{ti}^{*}\left[(\mathbf{q}_{t+\tau i}\mathbf{P}_{t+\tau}' + B_{t+\tau i}) \times \prod_{k=0}^{\tau}(1 + r_{t+k-1})^{-1}\right] \geq 0 \qquad \forall t \geq 0,$$

where $\mathbf{q}_{ti}\mathbf{P}_t' + B_{ti}$ is nonlabor wealth at time t and E_{ti}^{*} is the expectation under a probability measure, which is equivalent to the subjective probability measure of individual i used by individual i to evaluate the present value of future income streams.[4] Further discussion with references about the no-Ponzi-game condition can be found in the Appendix, which is available from the author, upon request.

A solution to individual i's problem must satisfy the Euler equation for bonds,

$$(9)\qquad \exp(-\gamma c_{ti}) = \frac{1 + r_t}{1 + \rho}\mathrm{E}_t[\exp(-\gamma c_{t+1i})];$$

the Euler equation for the I risky securities,

$$(10)\qquad \mathbf{P}_t\exp(-\gamma c_{ti}) = \frac{1}{1+\rho}\mathrm{E}_t[\exp(-\gamma c_{t+1i}) \times (\mathbf{P}_{t+1} + \mathbf{D}_{t+1})], \qquad t \geq 0;$$

[2] The only known closed-form solution to an individual's problem, with expected utility preferences that are increasing everywhere, for a discrete time intertemporal economy where there is investment in one-period bonds is for individuals with CARA utility, see Ricardo J. Caballero (1990). Paul Willen (1999) also studies such an economy with risky financial assets included.

[3] We assume that agents are not allowed to pursue doubling strategies. One can formulate precise conditions on trading strategies so that they are ruled out [see, e.g., Darrell Duffie (1996) for conditions in a continuous time economy]. It has been shown by Edward Omberg (1989) that for negative exponential utility, in the Black Scholes economy, pursuing a doubling strategy leads to negative infinite expected utility.

[4] As is standard in this literature, we assume that agents trade at equilibrium prices and there are no arbitrage opportunities. However, when markets are incomplete, there is not a unique way to evaluate one's borrowing opportunities. Thus we must define for each individual how to evaluate future income streams [see Michael Magill and Martine Quinzii (1994)].

and the following condition, so that the transversality condition is satisfied:

$$(11)\quad \lim_{\tau\to\infty} \mathrm{E}_t\left[u'(c_{t+\tau i})(\mathbf{q}_{t+\tau i}\mathbf{P}'_{t+\tau} + B_{t+\tau i}) \times \left(\frac{1}{1+\rho}\right)^{\tau}\right] = 0 \qquad \forall\, t \geq 0.$$

We can rewrite this condition, equation (11), as[5]

$$(12)\quad \lim_{\tau\to\infty} \mathrm{E}^*_{ti}\left[(\mathbf{q}_{t+\tau i}\mathbf{P}'_{t+\tau} + B_{t+\tau i}) \times \prod_{k=0}^{\tau} (1 + r_{t+k-1})^{-1}\right] = 0 \qquad \forall\, t \geq 0.$$

We conjecture that the equilibrium interest rate is constant through time, $r_t = r$, that the quantities demanded of the CLICs are constant through time, $\mathbf{q}_{ti} = \mathbf{q}_i$, that price equals X_{tn} discounted by the riskless interest rate $P_{tn} = X_{tn}/r$, $\{B_{ti}\}_0^\infty$ is a linear deterministic function of time for all i, and that consumption follows a random walk with a drift. These conjectures can be summarized as follows:

$$(13)\qquad r_t = r \qquad \text{for } t \geq 0,$$

$$(14)\qquad P_{tn} = \frac{X_{tn}}{r} \qquad \text{for } t \geq 0,\ n = 1, \ldots, N,$$

$$(15)\qquad \mathbf{q}_{ti} = \mathbf{q}_i \qquad \text{for } t \geq 0,$$

$$(16)\qquad B_{ti} = a_i + b_i t \qquad \text{for } t \geq 0,$$

where a_i and b_i are constants, and

$$(17)\qquad c_{ti} = c_{t-1i} + \mu_i + v_{ti} \qquad \text{for } t \geq 1,$$

where μ_i is a constant and v_{ti} is an error term with zero conditional expected mean and constant variance $v_{ti} \sim N(0, \sigma^2_{v_i})$. A disadvantage of our negative exponential utility model is that, for some individuals, consumption will eventually become negative; we assume that this will not occur until the distant future, and so disregard this problem. There is no tractable or simple model that avoids all problems of approximation.

Moreover, we conjecture that v_{ti} is jointly normally distributed with the innovation $\boldsymbol{\varepsilon}_t$ and uncorrelated with lagged values. We shall establish that these conjectures are consistent with general equilibrium by assuming them, and then checking consistency with conditions for individual maximization and with market clearing at all times.

Note that if the conjectures hold up and the interest rate r turns out to be a "small" number, then prices of the CLICs will make large swings from period to period relative to the change in income, reflecting the changed expected present value of incomes out to infinity, and thus large capital gains or losses on existing holdings of the CLICs. But the changes pose no crisis for individuals because they have no incentive or need to retrade the CLICs. The situation is somewhat analogous to that of homeowners who say that they do not care about the swings in the value of their homes because they will live in them forever. Despite the possibly large price changes, people are always in a very comfortable situation: our model represents people as, in effect, paying a regular insurance premium $-\bar{D}_n$, at all times, receiving a payment X_{tn} as a sort of insurance claim reflecting their changed economic circumstances since time 0. If they chose their investments in the CLICs right, offsetting the change in their income since time 0, they can forget about making any further adjustments in their investment positions.

Using the first-order condition (9), the conjectured interest rate process (13), and using the process for consumption (17), we find that

$$(18)\quad \mu_i = \frac{1}{\gamma} \ln\left[\frac{1+r}{1+\rho} \mathrm{E}_t \exp(-\gamma v_{t+1i})\right].$$

We now find the optimal investments of the individuals $\mathbf{q}_{ti}$. Using equations (9) and (13),

[5] For the general theory and discussion of a transversality condition equilibrium for infinite horizon economies, see Magill and Quinzii (1994).

we can rewrite the first-order condition (10) as[6]

$$(19) \quad \mathbf{P}_t(1+r) - \mathrm{E}_t[\mathbf{P}_{t+1} + \mathbf{X}_{t+1} + \bar{\mathbf{D}}] = -\gamma \operatorname{Cov}_t[c_{t+1i}, \mathbf{P}_{t+1} + \mathbf{X}_{t+1}].$$

Moreover, using our conjectured price processes $\mathbf{P}_t = \mathbf{X}_t/r$, and noting that $\mathrm{E}_t[\mathbf{X}_{t+1}] = \mathbf{X}_t$, we may rewrite equation (19) as

$$(20) \quad \bar{\mathbf{D}} = \gamma \operatorname{Cov}_t(c_{t+1i}, \mathbf{P}_{t+1} + \mathbf{X}_{t+1}).$$

To use equation (20) in order to find the individual's optimal investments $\mathbf{q}_i$, we need to solve for the innovation in consumption v_{ti}, to obtain an explicit solution for the covariance term. Using the budget constraint (7), along with our conjecture of constant investments (15), our conjectured bond demand (16), and our conjectured consumption process (17), we see that the innovation in consumption is linear in the innovation in income and in price, which is in turn linear in the innovation in income, and so

$$(21) \quad v_{ti} = \varepsilon_{ti} + \mathbf{q}_i\mathbf{A}'\boldsymbol{\varepsilon}_t \qquad \text{for } t \geq 1;$$

thus, our conjecture that v_{ti} is jointly normally distributed with $\boldsymbol{\varepsilon}_t$ is confirmed. v_{ti} will have a smaller variance than ε_{ti} because of the success of the risk-management contracts. Note that $v_{ti} \sim N(0, \sigma^2_{v_i})$, where

$$(22) \quad \sigma^2_{v_i} = \boldsymbol{\Sigma}_{ii} + \mathbf{q}_i\mathbf{q}_i' + 2\boldsymbol{\Sigma}_i\mathbf{A}\mathbf{q}_i',$$

where $\boldsymbol{\Sigma}_i$ is the ith row of $\boldsymbol{\Sigma}$ and Σ_{ii} is the ith diagonal element of $\boldsymbol{\Sigma}$.

Using equations (3), (14), (17), (20), (21), and (22) we obtain

$$(23) \quad \mathbf{q}_i = \frac{1}{\gamma}\frac{r}{1+r}\bar{\mathbf{D}} - \boldsymbol{\Sigma}_i\mathbf{A}.$$

Let $\mathbf{q}$ be an $I \times N$ matrix with $\mathbf{q}_i$ in the ith row. We can write $\mathbf{q}$ as

$$(24) \quad \mathbf{q} = \frac{1}{\gamma}\frac{r}{1+r}\boldsymbol{\iota}\bar{\mathbf{D}} - \boldsymbol{\Sigma}\mathbf{A},$$

where $\boldsymbol{\iota}$ is a $I \times 1$ vector of ones. This confirms our conjecture that $\mathbf{q}_t$ does not depend on t.

In equilibrium we must have $\boldsymbol{\iota}'\mathbf{q} = 0$, that is, these contracts are in zero net supply. Using this equilibrium condition and equation (24), we obtain the following equilibrium risk premia:

$$(25) \quad \bar{\mathbf{D}} = \frac{1}{I}\gamma\frac{1+r}{r}\boldsymbol{\iota}'\boldsymbol{\Sigma}\mathbf{A}.$$

If we substitute equation (25) into equation (24), we obtain $\mathbf{q} = -\mathbf{M}\boldsymbol{\Sigma}\mathbf{A}$, where $\mathbf{M} = \mathbf{I} - (1/I)\boldsymbol{\iota\iota}'$. $\mathbf{M}$ is symmetric and idempotent, $\mathbf{MM} = \mathbf{M}$, and $\mathbf{M}\boldsymbol{\iota} = 0$. For any vector $\mathbf{A}$ that defines the risky payment stream, and with the corresponding defined price processes, we have derived the optimal demands for individuals, and the equilibrium risk premia. We have also confirmed that the risk premia are constant over time, hence the name constant absolute risk premium (CARP) model. We still need to derive the equilibrium interest rate and the equilibrium borrowing and lending by individuals. We begin with the equilibrium borrowing and lending by individuals. To do this we take the budget constraint, equation (7), rearrange it, and use the constant investment conjecture, equation (15), to obtain

$$(26) \quad B_{t-1i} = (c_{ti} - y_{ti} - \mathbf{q}_i(\mathbf{X}_t + \bar{\mathbf{D}})') \times \left(\frac{1}{1+r}\right) + B_{ti}\left(\frac{1}{1+r}\right) \qquad \text{for } t > 0.$$

Equation (26) does not hold for time $t = 0$ because there are no dividend payments at time zero and because the interest rate prior to time zero, r_{-1}, need not equal r, the equilibrium interest rate from time zero on. Instead, the budget constraint at $t = 0$ is

$$(27) \quad B_{0i} = y_{0i} - c_{0i} + B^-_{0i},$$

where B^-_{0i} is the initial value of bonds at time 0 just before trade, equal to $B_{-1i}(1 + r_{-1})$. Once

[6] We use the fact that for any two random variables x and y, $\mathrm{E}(xy) = \mathrm{E}(x)\mathrm{E}(y) + \operatorname{Cov}(x, y)$ and, assuming joint normality, we use Stein's Lemma (see Charles M. Stein, 1981), that $\operatorname{Cov}(f(x), y) = \mathrm{E}(f'(x))\operatorname{Cov}(x, y)$.

we solve for c_{0i}, then from equations (17), (27), and (26) we can obtain the path for borrowing and lending for each individual i. If we take expectations conditional on information at time t and solve equation (26) forward, noting that $\lim_{\tau\to\infty} \mathrm{E}_t(B_{t+\tau i})[1/(1 + r)]^{\tau} = 0$, and substitute equation (17) into the resulting equation we obtain

$$(28)\quad B_{t-1i} = \left(\frac{1}{r}\right)^2 \mu_i + \frac{1}{r}(c_{ti} - y_{ti} - \mathbf{q}_i\mathbf{X}_t' - \mathbf{q}_i\bar{\mathbf{D}}') \quad \text{for } t > 0.$$

Substituting equation (27) into equation (28) for $t = 1$ and taking expectation at $t = 0$ and solving for c_{0i} we obtain

$$(29)\quad c_{0i} = y_{0i} + \frac{1}{1+r}\mathbf{q}_i\bar{\mathbf{D}}' - \frac{1}{r}\mu_i + \frac{r}{1+r}B_{0i}^{-},$$

and for $t \geq 1$,

$$(30)\quad c_{ti} = y_{ti} + \mathbf{q}_i\bar{\mathbf{D}}' + \mathbf{q}_i\mathbf{X}_t' + rB_{t-1i} - \frac{1}{r}\mu_i.$$

From (27) and (29) we have

$$(31)\quad B_{0i} = \frac{1}{r}\mu_i - \frac{1}{1+r}\mathbf{q}_i\bar{\mathbf{D}}' + \frac{1}{1+r}B_{0i}^{-}.$$

We can further show (see the Appendix) that

$$(32)\quad B_{ti} = B_{0i} + \frac{\mu_i}{r}t,$$

and thus, if μ_i is constant, we confirm our conjecture that bond demands are a linear deterministic function of time with $a_i = B_{0i}$, and $b_i = \mu_i/r$. We still need to derive what the value of μ_i is in equilibrium. We do this when we find the equilibrium interest rate. To solve for the equilibrium interest rate we note that all bond demands by all individuals must sum to zero or, by Walras' law, we can simply use the equilibrium condition in the goods market, that is,

$$(33)\quad \sum_{i=1}^{I} c_{ti} = \sum_{i=1}^{I} y_{ti}.$$

Recalling that $\Sigma_{i=1}^{I}\mathbf{q}_i = \mathbf{0}$ and $\Sigma_{i=1}^{I} B_{ti} = 0$, we have

$$(34)\quad \sum_{i=1}^{I} \mu_i = 0.$$

Now substitute equation (18) into (34) to obtain the equilibrium interest rate:

$$(35)\quad \ln(1+r) = \ln(1+\rho) - \frac{1}{2}\gamma^2\frac{1}{I}\sum_{j=1}^{I}\sigma_{v_j}^2.$$

The interest rate r will always be less than or equal to the subjective discount rate ρ, and strictly less so long as individuals still bear some risk in equilibrium. Note that the interest rate r_{-1} before the CLIC securities are constructed is given by the same expression but with $\sigma_{\varepsilon_j}^2$ in place of $\sigma_{v_j}^2$.

Now if we substitute equation (35) into (18) we obtain

$$(36)\quad \mu_i = \frac{1}{2}\gamma\left(\sigma_{v_i}^2 - \frac{1}{I}\sum_{j=1}^{I}\sigma_{v_j}^2\right),$$

which confirms our conjecture that μ_i is constant through time.

A. *Discussion*

The equilibrium in the constant absolute risk premium model can be summarized in just nine equations. Consumption is given by a simple difference equation (17), where the right-hand term v_{ti} is given by equation (21) and the term μ_i is given by equation (36). The initial condition for the difference equation, at time 0, is given by equation (29). The demand for the two kinds of securities is given by equation (32) for bonds [with bond demand at time 0 given by equation (31)] and equation (24) for the CLICs.

The risk premia are given by equation (25) and the interest rate is given by equation (35).

We can see from the consumption processes (17) and the error term (21) that one of the benefits from investing in the CLICs is the possible risk reduction in one's consumption process. Let us for the moment assume that there are no CLICs in the economy and only bonds are traded. Then, of course, one will be unable to reduce the riskiness of one's consumption process, though one is able to save some of his or her income in case there is a bad shock in the future. Because all of the shocks to income are permanent shocks there is no possibility of "smoothing" shocks over time through borrowing and lending. This particular motive for savings is the precautionary motive for savings.[7] Because transitory shocks can be smoothed away with borrowing and lending, CLICs would not be very useful if most of the shocks in the economy were transitory. It is the permanent component of the shocks that CLICs are useful to hedge.

To interpret this savings motive in this model we notice that the shock to consumption will be identical to the shock to income when there are no CLIC securities to trade the risks in the economy. As such, given the consumption process, we have that with only a bond market and no CLIC securities, the drift term μ_i will depend on one's own consumption risk, which in this case is equal to income risk, relative to the average consumption risk in the economy (36). In particular, individuals whose risk is larger than average will lend to individuals whose risk is less than average.

Our intuition tells us that this should be the case: individuals with a riskier consumption path will lend because there is a higher probability that a bad shock will be worse for them than it will be for someone with a less risky consumption path. We must keep in mind our assumptions that lead to this result; all individuals have the same coefficient of absolute risk aversion, all individuals have the same rate of time preference, the third derivative of the utility function is positive, and utility is additive separable. It is known that when utility is separable and the third derivative is positive and one cannot insure their income shocks, then an increase in the volatility of the income shocks will reduce consumption and increase savings today [see, e.g., Caballero (1990)].

Now, once CLIC markets are included in the model, what will change is the variance of the consumption process and the drift term in the consumption process (because it is a function of the variance in the consumption process), as well as consumption at time zero. Thus when there is only a bond market, an individual in the economy may be a lender because he or she is riskier than average, whereas the same individual in the economy when there is a bond market and one CLIC market may be a borrower if his or her consumption process is less risky than average after taking positions in the CLIC market. The reason why this may occur is that by taking positions in the CLICs, if the CLICs are well correlated with one's endowment process, then one will be able to hedge much of his or her risk. As such, if one lays off much of his or her risk, the precautionary motive for saving (lending) declines.

We also see that consumption at time zero changes when we include CLIC markets, not only because the precautionary motive for saving changes, μ_i, but also because we expect future payments from the CLIC securities as dividends (29), $[1/(1 + r)]\mathbf{q}_i\bar{\mathbf{D}}'$ and interest payment from bonds at the new interest rate. When we take positions in the available CLICs, we expect at time zero to receive from time $t = 1, \ldots, \infty$ a dividend stream equal to the risk premium, which is $\mathbf{q}_i\bar{\mathbf{D}}'$. This part of the dividend component is riskless. The individuals will also consume the risky part of their dividends as well. To smooth the riskless part of one's dividend stream over an entire lifetime, individual i consumes $[1/(1 + r)]\mathbf{q}_i\bar{\mathbf{D}}'$ of it in each period of his or her lifetime. Consumption at time zero also changes because of the change in the interest rate, the last term in equation (29), $[r/(1 + r)]B_{0i}^{-}$. B_{0i}^{-} is the value of individual i's bonds held over from $t = -1$ at time zero. Given that the interest rate changes from r_{-1} to r, we expect future interest payments on these bonds from $t = 1, \ldots, \infty$ to be rB_{0i}^{-}. To smooth this over one's lifetime, individual i consumes

[7] Caballero (1990) covers this precautionary motive for savings for an individual when income follows general ARMA processes.

$[1/(1 + r)]rB_{0i}^{-}$ of it in each period of his or her lifetime.

Thus the savings decision is affected by the new CLIC securities. From equations (32) and (31) we see that savings decisions (buying bonds) will depend on the precautionary motive μ_i, and on a "smoothing" motive, the second and third term on the right-hand side in equation (31). Let us look at the demands by individuals for the CLICs (24). We see that an individual will demand more of a CLIC the higher its risk premium $\bar{\mathbf{D}}$. An individual will demand more of a CLIC the less the CLIC covaries with the individual's endowment, where $\boldsymbol{\Sigma}_i\mathbf{A}$ is the row vector of covariances of the CLICs with the individual's endowment. This is as we expect: the higher the expected payoff of a security, the higher will be the demand for that security. The higher the hedging services of a security, the more attractive is the security to an investor and the more the investor will demand.

The risk premia will depend on the covariance of the aggregate consumption processes with the values of the CLIC security next period [see equation (20)] and aggregate over $i = 1, \ldots, I$. This is a standard result in intertemporal capital asset pricing, which was first clearly exposited by Douglas T. Breeden (1979); the generality and robustness of this result was indicated by Sanford J. Grossman and Shiller (1982). Our conjectured processes for prices and the risky component of dividends ensures that the risk premia $\bar{\mathbf{D}}$ will be constant over time. In particular we have devised securities in which individuals will take positions at time zero, consume the dividends forever, and never rebalance portfolios. Given these prices and dividends, individuals will always maintain the same investment positions in the CLICs.

The equilibrium interest rate will depend in part on the rate of time preference in the economy (35). One way to see how is to take a world where there is no risk. Then the interest rate will equal the rate of time preference and there will be no borrowing or lending in the economy. We can see this because the drift term will necessarily be zero in the consumption processes for all individuals and, thus, the consumption and income processes will coincide in this case and both will be riskless. Given that the rates of time preferences are the same across individuals, no one will want to borrow or lend because of patience or impatience in this world, in that we have effectively assumed this away by assuming the same rate of time preference across individuals.

Once we introduce risk in the economy, the interest rate will still depend on the rate of time preference but will also depend on the average risk of consumption in the economy. As an example, take a world where there is no risk in the aggregate but individuals have some risk in their endowment. Then in a world with only a bond there is still a motive to save for precautionary reasons and the average risk in the economy will also determine the interest rate. If there were CLICs in the economy that allowed all individuals to hedge their income risk perfectly, then the interest rate would once again equal the rate of time preference and there would once again be no borrowing or lending.

If we are in a world where there is some aggregate risk, that is, market risk, then the interest rate will always be less than the rate of time preference. Even if everyone is perfectly hedged, the interest rate could never equal the rate of time preference. The reason for this is that, as long as all individuals have some risk in their consumption processes, they will have a precautionary motive for saving. Thus if the interest rate equaled the rate of time preference, and all individuals held the market portfolio (world portfolio), then all individuals would want to save as a precautionary motive. However, the market for bonds would not clear because all individuals would demand bonds. Thus the interest rate would have to be lower than the rate of time preference to offset this precautionary motive for savings to clear the bond market.

III. Contract Designer's Problem: The Risk-Optimal Interest Rate

Let us now turn to the contract designer's problem, which is to define a small number, $N \ll I$, of optimal securities, WICs, and to show the designer wishes to maximize the interest rate. We assume that the contract designer wishes to choose $\mathbf{A}$ to maximize a social welfare function, which we assume is the negative sum of the log of negative lifetime expected utilities over all individuals, where the individual's utilities are maximized, given equilibrium prices $\mathbf{P}_t$, dividends $\mathbf{D}_t$, and interest rate r. The contract designer maximizes

$$S_0 = -\sum_{i=1}^{I} \ln(-U_{0i}), \tag{37}$$

subject to

$$\mathbf{A}'\mathbf{\Sigma}\mathbf{A} = \mathbf{I}. \tag{38}$$

By assuming that the social welfare function is loglinear, we achieve much simplification of expressions. It is also the social welfare function that results in the same contract design problem as in the one-period mean-variance case under similar assumptions.[8]

To solve the problem for the contract designer, we must derive an expression for the lifetime expected utility for the individuals in the economy. We can show (see the Appendix for derivation) that lifetime expected utility at time zero U_{0i} can be expressed as

$$U_{0i} = -\frac{1+r}{r}\exp(-\gamma c_{0i}), \tag{39}$$

and so, using equations (33) and (39) and substituting into equation (37) we can rewrite the social welfare function as

$$S_0 = -\sum_{i=1}^{I} (\ln(1+r) - \ln(r) - \gamma y_{0i}). \tag{40}$$

One will notice that the social welfare function is monotonically increasing in r. Thus to define new contracts all we need to do is to maximize r or $\ln(1 + r)$ with respect to $\mathbf{A}$, subject to the normalization constraint (38). Equivalently, using equation (35), we can maximize $\text{tr}(\mathbf{q}'\mathbf{q})$ or, in terms of the $\mathbf{A}$ matrix, we can maximize the expression[9]

$$S_1 = \text{tr}(\mathbf{A}'\mathbf{\Sigma}\mathbf{M}\mathbf{\Sigma}\mathbf{A}), \tag{41}$$

where we have defined $\mathbf{M} = \mathbf{I} - (1/I)\boldsymbol{\iota\iota}'$.

We set up the Lagrangian that represents the constraint that diagonal elements of $\text{Var}(\mathbf{X}_t) = \mathbf{A}'\mathbf{\Sigma}\mathbf{A}$ equal 1 and off-diagonal elements equal 0. The Lagrangian is

$$L = \text{tr}(\mathbf{A}'\mathbf{\Sigma}\mathbf{M}\mathbf{\Sigma}\mathbf{A}) - \sum_{n=1}^{N}\sum_{m=1}^{n} (\mathbf{A}_m'\mathbf{\Sigma}\mathbf{A}_n - e(m, n))\lambda_{mn} \tag{42}$$

where

$$e(m, n) = \begin{cases} 0 & \text{when } m \neq n \\ 1 & \text{when } m = n. \end{cases} \tag{43}$$

First-order conditions for a maximum are

$$\frac{\partial L}{\partial \mathbf{A}_n} = \mathbf{\Sigma}\mathbf{M}\mathbf{\Sigma}\mathbf{A}_n - \mathbf{\Sigma}\mathbf{A}_n\lambda_{nn} - \sum_{n=1}^{N}\sum_{m=1}^{n} \mathbf{\Sigma}\mathbf{A}_m\lambda_{mn} = 0, \quad n = 1, \ldots, N; \tag{44}$$

$$\frac{\partial L}{\partial \lambda_{nn}} = \mathbf{A}_n'\mathbf{\Sigma}\mathbf{A}_n - 1 = 0, \quad n = 1, \ldots, N; \tag{45}$$

and

$$\frac{\partial L}{\partial \lambda_{mn}} = \mathbf{A}_n'\mathbf{\Sigma}\mathbf{A}_m = 0, \qquad m \neq n. \tag{46}$$

In this case, we need consider explicitly only the diagonal constraints in $\mathbf{A}'\mathbf{\Sigma}\mathbf{A} = \mathbf{I}$ because the off-diagonal elements will be zero even if unconstrained. This particular result is shown in

[8] In social welfare theory, it is normally assumed that the social welfare function is an increasing and concave function of individual utilities [see, e.g., Andrew Mas-Colell et al. (1995)]. Ours being convex in individual utilities, and utilities being negative exponential, effectively offsets diminishing marginal utility and results in our maximizing risk sharing, that is, minimizing consumption variance, without regard to income levels. In practice when we experimented with concave social welfare functions, optimal contracts were in effect designed for people in the poorest regions of the world. These are the regions where we suspect that innovative risk-sharing institutions are least likely to succeed in practice.

[9] To obtain this particular expression see the Appendix.

J. N. Darroch (1965) and in Masashi Okamoto and Mitsuyo Kanazawa (1968). In matrix form, the first-order conditions reduce to

$$\mathbf{M\Sigma A} = \mathbf{A\Lambda}, \tag{47}$$

where $\mathbf{\Lambda}$ is a diagonal matrix whose nth diagonal element is λ_{nn}. Thus, the columns of the desired matrix $\mathbf{A}$ are determined as N eigenvectors of the matrix $\mathbf{M\Sigma}$, whose ijth element is the covariance between the deviation of individual i's income from world-average income and individual j's income. To find which eigenvectors should be included in $\mathbf{A}$, we premultiply (47) by $\mathbf{A'\Sigma}$ and using $\mathbf{A'\Sigma A} = \mathbf{I}$, find that $\mathbf{A'\Sigma M\Sigma A} = \mathbf{\Lambda}$, which is diagonal. Comparing this with (41) we find that the objective function equals the sum of the eigenvalues of the included eigenvectors. Thus the social planner chooses the N eigenvectors with the largest eigenvalues to define the WICs. We will arrange the columns of $\mathbf{A}$ in order of decreasing eigenvalues, so that the more important contracts are toward the left.

We now show that all of our optimal WIC securities will be essentially swaps. Postmultiplying (47) by $\mathbf{\Lambda}^{-1}$, one finds that $\mathbf{A} = \mathbf{M\Sigma A\Lambda}^{-1}$. Given that $\mathbf{M}$ is idempotent, $\mathbf{MA} = \mathbf{MM\Sigma A\Lambda}^{-1} = \mathbf{A}$. It follows, given that $\boldsymbol{\iota}'\mathbf{M} = \mathbf{0}$, that $\boldsymbol{\iota}'\mathbf{A} = \mathbf{0}$. Thus, the sum of the elements in each column of $\mathbf{A}$ equals zero.

Given that $\mathbf{MA} = \mathbf{A}$, we can rewrite (47) as

$$\mathbf{M\Sigma MA} = \mathbf{A\Lambda}. \tag{48}$$

$\mathbf{A}$ is just the matrix of the N eigenvectors of $\mathbf{M\Sigma M}$ corresponding to the highest eigenvalues. $\mathbf{M\Sigma M}$ is the variance matrix of deviations of individual endowments from the world endowment. If we substitute equation (25) into equation (24), we obtain $q = -\mathbf{M\Sigma A}$. Because we have that $\mathbf{q} = -\mathbf{M\Sigma A}$, from (47) we also see that $\mathbf{q} = -\mathbf{A\Lambda}$. Note also that, because $\mathbf{A}$ is an eigenmatrix of a symmetric positive-semi-definite matrix, $\mathbf{A'A}$ is a diagonal matrix. Given that $\mathbf{A'\Sigma A} = \mathbf{I}$, the diagonal elements of $\mathbf{A'A}$ are inversely proportional to the corresponding eigenvalues.

We can now, using the risk-optimal interest rate model, produce measures that place a dollar value on the availability of these WIC securities. The amount of (certain) endowment increase per period for individual i to be as well off without the N WIC securities as with the N WIC securities is given by F_{iN}. Define c_{ti}^{-} as the consumption stream for individual i if WIC securities were never constructed, and only a bond market exists. Also, define r_{-}, in contrast to the risk-optimal interest rate, as the interest rate that would obtain if there were no WIC securities; it is the interest rate prior to the construction of the new WIC securities. Then F_{iN} is the solution to

$$\mathrm{E}_0\left[\sum_{\tau=0}^{\infty} u_0(c_{\tau i}^{-} + F_{iN})/(1+\rho)^{\tau}\right] = \mathrm{E}_0\left[\sum_{\tau=0}^{\infty} u_0(c_{\tau i})/(1+\rho)^{\tau}\right], \tag{49}$$

where F_{iN} is a constant. We can solve for F_{iN} as[10]

$$F_{iN} = (c_{0i} - c_{0i}^{-}) + \frac{1}{\gamma}(\ln(r) - \ln(1+r) - \ln(r_{-}) + \ln(1+r_{-})). \tag{50}$$

We see that the welfare gain measure depends on two terms, the first of which is the change in time zero consumption from before to after risk sharing. The second term takes into account the change in the interest rate, which will be positive here, given that $r > r_{-}$. Note that for some individuals the first terms will necessarily have to be negative so that the goods market will clear at time zero.

One of the interesting results in this section is that the contract design problem (under the assumption that the welfare function, which is the sum of the negative logs of negative expected utilities), results in the contract design problem that is identical to that in a one-period mean variance world, where agents have identical coefficients of absolute risk aversion [compare equation (47) with Athanasoulis and Shiller

[10] See Appendix for derivation.

(2000) Theorem 1]. As such, we are able to make simple prescriptions on how optimal contracts are to be chosen. One estimates a variance matrix and uses equation (47) to choose the **A** matrix. One will notice that under the assumptions in this model, if the contract designer were allowed to change the **A** matrix after time zero, the designer would not because that same **A** matrix would still be optimal. Thus the contracts designed here are time consistent.

IV. An Application: The G-7 Countries

As an example of our methods we apply it to the G-7 countries: Canada, the United States, Japan, France, Germany, Italy, and the United Kingdom. These are representative of the developed world where innovative risk management contracts are most likely to succeed today. To apply our methods we need estimates of the variance matrix $\mathbf{\Sigma}$, the real risk-free interest rate r, the coefficient of absolute risk aversion γ, and the rate of time preference ρ.

A few points are very important in the understanding of the results below. First, the variance matrix is the only parameter that will affect the WIC securities and the optimal demands for the WIC securities. The choice of the interest rate, the coefficient of absolute risk aversion, and the rate of time preference will have no effect on these, although they will affect the welfare gains. In that we are able to formulate the contract design problem as, in effect, a one-period problem, discounting does not matter in the estimates of contracts, although clearly discounting will affect the contracts' worth to individuals.

A. *A Three-Level Income Model of Individual Income*

We are able to estimate the matrix Σ for individuals in these countries, even though its dimensions are very large (the combined population of these countries in 1992 was 644,594,000), by assuming that all individuals within a country have the same income (endowment) process and using a model of the changes in incomes of individuals that implies that the entire matrix is determined by four parameters. Our three-level income model represents the change in individual i's income from $t-1$ to t is taken to be the sum of three components representing three levels of income comovement:

$$(51) \qquad \Delta y_{ti} = \varepsilon_{ti} = u_{tw} + u_{tc} + u_{ts},$$

where u_{tw} is a world shock, common to all people in the world; u_{tc}, $c = 1, \dots, 7$, is a country shock, common to everyone in country c but uncorrelated with the country shock of individuals in other countries; and u_{ts}, $s = 1, \dots, 7$, is a spatial shock that is common to everyone in country s but correlated with the spatial shock of individuals in other countries according to a spatial model.[11] Each of the three shocks is assumed to be normally distributed with zero mean and constant variance through time, serially uncorrelated, and uncorrelated with the other two shocks. Note that the expected changes to income is zero, which is also the case in the estimation.

With this method of estimating $\mathbf{\Sigma}$, we impose prior restrictions that all individuals have the same mean (zero) and variance of changes in real per capita income (endowment), and that covariances are determined by a spatial component of risk between countries and a common component of risk. These prior restrictions make it possible to estimate a sensible variance matrix with limited data, one that represents world, idiosyncratic, and spatial shocks, but that otherwise represents countries symmetrically.

Our prior assumptions for estimating the variance matrix of changes in real per capita national incomes are represented by the following formulas for the elements of $\mathbf{\Sigma}$:

$$(52) \qquad \sigma_{i,i} = \exp(\alpha^{w}) + \exp(\alpha^{c}) + \exp(\alpha^{s}) \qquad i = 1, \dots, I$$

$$\sigma_{i,j} = \exp(\alpha^{w}) + \exp(\alpha^{s})\exp(-\delta d_{i,j}) \qquad i \neq j,$$

[11] If we were to expand this model to a four-level income model by adding an individual specific shock, that is, a shock specific to each individual in each country, the results of this section would be unchanged. From equation (47) and using the fact that $\mathbf{MA} = \mathbf{A}$, one can see that if we add an identity matrix times a constant to Σ, then the equations are solved by the same $\mathbf{A}$ matrix, whereas the eigenvalues are each increased by an amount equal to the constant.

where d_{ij} is the distance between countries (individuals) i and j, measured as air miles between the major city in the respective countries. We used the air mile distances between the major cities Montreal, New York, Tokyo, Paris, Berlin, Rome, and London. The parameters to be estimated are α^w, α^c, and α^s, which are associated with the world, country, and spatial components, and δ. The correlation between the spatial shock of individuals in country i with the spatial shock of individuals in country j is $\exp(-\delta d_{i,j})$. Because δ is positive, the farther away the major city of two countries, the less is the covariance of per capita income between the two countries. This formula corresponds to a valid (i.e., the variance matrix is nonnegative definite for any placement of cities) isotropic (i.e., the model is invariant to rotations of the coordinate system) spatial model where the cities lie in $\mathbb{R}^2$ [see Noel Cressie (1991 p. 86)]. The formula also corresponds to a valid isotropic spatial model where the cities lie on the surface of a sphere and distances are measured along great circles, as in our application to the earth. Moreover, the variance matrix is strictly positive definite unless two cities coincide.

This formulation restricts all covariances to be positive. The prior restriction that all covariances are positive may seem strong, but it is maintained here as a sort of commonsense prior notion that there is really no reason in general for any pairs of countries to tend to move opposite each other. This restriction may serve to reduce the possibilities for diversification, by eliminating the negative correlations that diversifiers seek. The effect of the restriction will tend to be to make it more difficult to make a case for the WIC securities.

We estimate all parameters in the variance matrix with data on per capita gross domestic products in 1985 U.S. dollars from the Penn World Table [see Summers and Heston (1991) updated to 1992 from http://www.nber.org]. A maximum likelihood estimate is taken using the 42 observations on per capita income changes for each country. Future work will examine similar models of the variance matrix with other "commonsense" priors.

B. *Calibration of r, γ, and ρ*

In this section we specify the real risk-free interest rate r and the coefficient of absolute risk aversion γ. Once we specify these two parameters, the subjective rate of discount ρ is determined by equation (35). Empirical studies have found wildly different estimates of the coefficient of relative risk aversion parameter (and the discount rate), depending on the kind of circumstances that generate the data [see Richard H. Thaler (1990)]. Values of the coefficient of relative risk aversion have been estimated in the 100's, but these may be regarded as implausibly high; we choose it equal to three as representing a sort of consensus by many who work in this literature as a reasonable value to assume. To obtain the coefficient of absolute risk aversion γ, we take the average income in 1992, $\bar{y} = 14783.43$ (see Table 1) and setting $\gamma = 3/\bar{y}$, we obtain $\gamma = 0.000203$.

To obtain an estimate of r, we use the data from J. Huston McCulloch and Heon-Chul Kwon (1993) for one-month maturity zero coupon bonds for the whole sample, 1946:12–1991:2 for the nominal interest rate. To obtain an inflation series we use the CPI-U not seasonally adjusted from the U.S. Bureau of Labor Statistics (BLS). We take the geometric average of the real interest rates and obtain $r = 0.49$ percent as the annualized real interest rate. With these values of r and γ we obtain from equation (35) that $\rho = 0.0077$.

C. *Results*

The estimated parameters of (52) are $\hat{\alpha}^w = 11.05$, $\hat{\alpha}^c = 9.88$, $\hat{\alpha}^s = 10.82$, and $\hat{\delta} = 0.00017$. These imply a standard deviation of annual per capita income change of \$364.60, and a standard deviation of the 25-year income change of \$1,823. Of the three components, the most variable is the world component, which has no effect on our WICs because the world component cannot be hedged away. The spatial component is somewhat more important than the country-specific component: its standard deviation is 1.6 times larger. The estimate of δ implies that the correlation between the spatial components of the United States and Canada (320 miles) is 0.95, whereas the correlation between the spatial components of the United States and Japan (6740 miles) is 0.32.

We show, for $N = 2$, the elements of $\mathbf{A}$ for individuals in each country in Table 1. Because each individual within a country gets the same

TABLE 1—POPULATION, 1992 GDP PER CAPITA, OPTIMAL CONTRACTS, AND OPTIMAL INVESTMENTS

Country	1992 population (in 000's)	1992 GDP (per capita)	$\mathbf{A}_{i1}$ ($\times 10^{-9}$)	$\mathbf{A}_{i2}$ ($\times 10^{-9}$)	$\mathbf{q}_{i1}$	$\mathbf{q}_{i2}$
Canada	27,445	\$16,362	6.04	−0.60	−58.44	3.99
			(1.32)	(0.84)	(17.2)	(5.2)
United States	255,000	\$17,945	14.62	−4.40	−141.48	29.41
			(1.05)	(3.23)	(9.5)	(19.2)
Japan	124,000	\$15,105	−16.33	−24.21	157.97	161.81
			(3.21)	(14.03)	(27.7)	(79.7)
France	57,372	\$13,918	−7.41	17.79	71.67	−118.94
			(2.27)	(9.64)	(22.9)	(54.1)
Germany	65,120	\$14,709	−8.54	17.24	82.59	−115.24
			(1.73)	(9.37)	(18.5)	(52.3)
Italy	57,809	\$12,721	−8.34	17.33	80.71	−115.86
			(1.61)	(9.09)	(17.6)	(51.3)
United Kingdom	57,848	\$12,724	−7.03	17.20	68.06	−114.94
			(2.24)	(9.33)	(22.4)	(52.3)

Notes: The $\mathbf{A}_{in}$ columns, $n = 1, 2$, give the ith element of $\mathbf{A}_n$ if individual i is in the country shown in the leftmost column. The $\mathbf{A}_{in}$ elements shown therefore give the weight given to the income of each individual in a country in determining the dividend paid on one contract n. The numbers in the optimal investment columns $\mathbf{q}_{in}$, $n = 1, 2$, are the numbers of contracts n individual i will buy according to the theory if the individual is in the country shown in the leftmost column. \$ denotes 1985 U.S. dollars. The risk premia are $\bar{D}_1 = \$1.76$ when one WIC is constructed and $\bar{D}_1 = \$1.69$ and $\bar{D}_2 = \$0.41$ when two WICs are constructed in 1985 dollars. Standard errors, in parentheses, were obtained with Monte Carlo methods.

weight, we need not display all 644,594,000 rows of **A**: we show only one of the rows for each country. Looking at the first column of the **A** matrix, in Table 1 we see that the first contract is essentially a U.S. versus Japan swap. This contract weights individuals in the United States and individuals in Japan in the opposite direction with the highest weights. In this contract there are also weights on the core European Union (Core-EU) countries, which we define as France, Germany, Italy and the United Kingdom, and these are on the same side as Japan with similar weights, although they are about half that of Japan's. Canada has a positive weight, as does the United States, with less than half the weight that is given to U.S. individuals. As such we expect that the U.S. and Japanese individuals will gain the most benefit from this contract.

From the second column of **A** in Table 1 we see that contract 2 is essentially a Japan versus (Core-EU) swap. The two contracts also span, approximately a U.S. versus (Core-EU) swap. This can be achieved by taking a long position in the first WIC and about two-thirds of a short position in the second.

With these two WIC contracts it is possible for the United States, Japan, and the Core-EU as a group, to achieve "most" of the risk sharing possible between them. We see in the next two columns of Table 1, the first two columns of the **q** matrix show that the largest investment positions in the first WIC are taken by individuals in the United States and in Japan. The United States takes a negative position in the contract because it has a positive weight in the contract, so that it sells off some of its own income (endowment) risk. Japan on the other hand takes a positive position because it has a negative weight in the WIC. Thus the individuals in the United States pay a premium when they take positions in the WIC, given that we normalize $\bar{D}$ to be positive, whereas individuals in Japan receive a premium. The investments by the Core-EU are about half that of Japan and for Canada about one-third that of the United States. One can see similar results in the second column of **q** with the United States and Canada being insignificant in that WIC.

As mentioned earlier, only the variance matrix matters for contract definition and investment positions in the WIC securities. As such there are two effects that are driving these results. One is the differences in populations, whereas the second is the correlation between countries. One way to investigate this is to look at the WICs if the variance matrix: (a)

TABLE 2—WELFARE GAINS IN DOLLARS AND AS PERCENT OF 1992 CONSUMPTION

Country	F_{i1}	F_{i2}	F_{i6}	F_{i1}/C_0^- (percent)	F_{i2}/C_0^- (percent)	F_{i6}/C_0^- (percent)
Canada	\$ 75.59	\$ 88.76	\$688.57	0.46	0.54	4.21
	(\$ 40.26)	(\$ 41.99)	(\$ 54.84)	(0.25)	(0.26)	(0.34)
United States	\$398.85	\$415.12	\$422.94	2.22	2.31	2.36
	(\$ 51.28)	(\$ 47.53)	(\$ 48.55)	(0.29)	(0.26)	(0.27)
Japan	\$495.00	\$980.96	\$968.95	3.28	6.49	6.41
	(\$155.9)	(\$127.4)	(\$125.2)	(1.03)	(0.84)	(0.83)
France	\$109.11	\$385.25	\$706.13	0.78	2.77	5.07
	(\$ 73.09)	(\$ 67.14)	(\$ 66.76)	(0.53)	(0.48)	(0.48)
Germany	\$141.91	\$400.53	\$718.99	0.96	2.72	4.89
	(\$ 64.24)	(\$ 59.62)	(\$ 72.15)	(0.44)	(0.41)	(0.49)
Italy	\$135.92	\$397.43	\$775.28	1.07	3.12	6.09
	(\$ 58.52)	(\$ 60.04)	(\$ 75.71)	(0.46)	(0.47)	(0.60)
United Kingdom	\$ 99.28	\$358.27	\$696.95	0.78	2.82	5.48
	(\$ 68.71)	(\$ 62.88)	(\$ 68.89)	(0.54)	(0.49)	(0.54)

Notes: F_{i1}, F_{i2}, and F_{i6} are welfare gains in 1985 U.S. dollars each year if 1, 2, or 6 WICs are created. Under our assumptions, with 6 WICs risk sharing is complete, and so F_{i6} represents the total possible welfare gain from risk sharing. F_{i1}/C_0^-, F_{i2}/C_0^-, and F_{i6}/C_0^- are welfare gains in 1992 as a percent of 1992 consumption. For these calculations we used $r_- = 0.49$ percent, $\gamma = 0.0002029$, and consequently $\rho = 0.0077$. The equilibrium interest rate when one, two, and six WICs are constructed is 0.52, 0.54, and 0.56 percent, respectively. Standard errors, in parentheses, were obtained with Monte Carlo methods.

had only a world component and country component and (b) had only a world component and a spatial component. In case a, given the model for the preceding variance matrix, what would drive the results is the differences in populations across countries. In general the first eigenvector will pick up, roughly speaking, a swap between the country with the largest population versus all other countries; the second will be a swap, roughly speaking, between the country with the second largest population and the remaining countries with smaller populations and so on. Thus in contract one, the United States and Canada would enter with opposite signs because Canada has the smallest population and the United States the largest. In case (b), where spatial correlations drive the results, one finds that the United States and Canada in the first and second contracts have very similar (positive) weights with the same sign. The other countries have similar (negative) weights as in the contracts in case a. Thus one may ask why the United States and Canada in Table 1 have such different weights: it is because, when comparing them according to spatial correlations, they are the most similar countries but comparing them according to populations, they are the most different countries.

We could extend our estimated variance matrix to include some of the smaller EU countries, Belgium or Denmark for example, and using their distances to fill out the variance matrix with the estimated parameters. Then the incomes of individuals in these countries would receive smaller weights than do the Core-EU countries in the columns of the **A** matrix, just as individuals in Canada get smaller weights than do individuals in the United States. As such these smaller EU countries would not be significant in the first few contracts.

The welfare gain for the countries involved are given in Table 2. The first three columns of numbers report the endowment increase needed each period to make individuals as well off without risk sharing as they would be with risk sharing using N contracts. The last three columns report the first three columns as a percentage of consumption that would have been realized at time zero if no risk sharing occurred.[12]

[12] Note that adding securities need not increase the welfare for each country: for Japan, F_{i6} is less than F_{i2} in Table 2. Adding WIC securities does not always raise social welfare in each country. Adding securities changes interest rates and risk premia in existing markets, which may have an adverse effect on individuals in some countries even though risk-sharing opportunities are increased.

The welfare gains are quite substantial. Notably, creating the first contract alone creates a welfare gain for Japan of $495.00 per capita and for the United States of $398.85 per capita, each year in perpetuity. The reason for the large welfare gain is that the United States and Japan are geographically far apart, and hence the variance matrix has them little correlated with each other, so there is substantial opportunity for risk sharing between them. The random walk assumption means that these independent year-to-year discrepancies between U.S. and Japanese income levels accumulate and may result in large differences in standards of living between the two countries, in the absence of the risk-sharing contract.

Creating the second contract increases the welfare gain for Japan by another $485.96, but has little extra benefit to the United States, which attains an additional welfare gain of only $16.27. The second contract is, in effect, designed to exploit the large geographical distance between Japan and Europe, which is nearly as great as the distance between Japan and the United States. Because of the relative geographic proximity between the United States and the Core-EU, the contract designer effectively left the United States out of this contract. Although the Core-EU is substantially correlated with the United States, the United States can derive little extra benefit from a (Core-EU)–Japan swap, given that it already has essentially a U.S.–Japan swap, and so those in the United States buy very little of this contract and hence achieve little welfare gain from it.

Welfare gains for the other countries from the first two contracts are also substantial, around $400 for each country except Canada. Canada achieves less benefit than the other countries from the first two contracts. Because of Canada's relatively small population, the contract designer effectively gives relatively little weight to Canada in designing the first two WICs, and thus these offer little opportunity for Canadians to hedge their country-specific shocks and achieve little welfare gain for Canada.

D. *Discussion of Possible Biases in the Calibration Results*

The results presented here are meant to be illustrative and plausible as a first pass. But we are aware of possible weaknesses in our assumptions and of possible arguments for other calibration parameters. In the risk-sharing literature, there is a wide variety of assumptions about the risk-free rate, the risk-adjusted growth rate, the parameters of risk aversion, and the measures of risk [see, e.g., Eric van Wincoop (1999)]. There are also differences in functional form of the utility function: a CRRA utility function is often assumed instead of our CARA utility, and some studies use nonexpected utility preferences to separate the coefficient of relative risk aversion from the elasticity of intertemporal substitution, such as those found in Larry G. Epstein and Stanley E. Zin (1989) and Philippe Weil (1990).

Given the wide variety of assumptions, it is difficult to summarize the reasons for differences between our results and theirs. Still, some exercises with our model in which we adjust some of our calibration parameters to values found in other studies are suggestive. We focus here on the impact of these assumptions on welfare gains, not on the contract definition as embodied in the **A** matrix. In our model, the **A** matrix depends only on the estimated one-period variance matrix and not on the parameters of the utility function.

For brevity, we will single out for comparison one study by Karen K. Lewis (2000) that uses the same data set and the same set of countries as ours, but that arrives at lower estimates of the welfare gains from risk sharing. Because her study, as well as many others, stresses the welfare gains from full risk sharing (i.e., creating all six markets in this paper), we will be comparing our welfare results for creating all six markets to her numbers. That is we compare the column F_{i6}/c_0^- in Table 2 with her Table 2, the column where $\gamma = \theta = 2$ and all countries have the same mean growth rate.[13] The unweighted average of the welfare gains in Lewis (2000) is 0.59 percent. In our Table 2 the unweighted average of the welfare gain across countries is 4.93 percent, a much higher welfare gain.

Her model appears different from ours in

[13] Because she uses Epstein and Zin (1989) preferences, it becomes CRRA preferences with a coefficient of relative risk aversion of 2 when $\gamma = \theta = 2$.

several dimensions. Hers is based on assumptions of CRRA utility (when she parameterizes the utility function with $\gamma = \theta = 2$), a nonzero growth rate (the same for all countries), and she uses consumption rather than income for her endowment process. Her shocks are persistent in log endowment (the log endowment follows a random walk with a drift), whereas ours are persistent in levels. We still believe that there are some basic similarities that make it possible to compare our results with hers if we align some parameters with hers. Although we do not have expected growth in our model, if we match the risk-free interest rate in our model with her appropriate overall discount rate (risk-free rate less risk-adjusted growth rate) and/or we match the coefficient of relative risk aversion, the comparison is reasonable.

In our parameterization for Table 2 we used a coefficient of relative risk aversion of 3, whereas Lewis assumed 2. Lowering this parameter in our model to 2, and making no other changes in our calibration, our revised Table 2 gives an unweighted average welfare gain across countries of 3.42 percent. The welfare gain has been reduced by 1.51 percent simply by lowering the coefficient of relative risk aversion. We think, however, that the higher coefficient of relative risk aversion is more in accord with the evidence. There are some studies that find very high coefficients of relative risk aversion, much higher than the value of 3 implicitly assumed here [see Robert B. Barsky et al. (1997)].

In our parameterization, we used a risk-free rate of 0.49 percent, whereas in Lewis the risk-free rate is 6.6 percent. Because in Lewis there is growth in the economy, we should subtract the risk-adjusted growth rate from her risk-free interest rate to obtain a comparable interest rate for an economy with no growth.[14] Her risk-adjusted growth rate is 2.3 percent and thus a comparable interest rate for an economy with no growth is 4.3 percent. Increasing our risk-free interest rate from 0.49 to 4.3 percent, while keeping the coefficient of relative risk aversion at 2, results in an unweighted average welfare gain of 0.40 percent. This is a further reduction of the welfare gain by over 3 percent. Thus, even though our shocks are persistent, if we discount the future enough, the welfare gain today of these future shocks will not be as important.

Much of the discussion of the parameterization of the utility function is intimately tied to the equity premium puzzle [see R. Mehra and E. C. Prescott (1985)] and the risk-free rate puzzle [see Weil (1989)]. When parameterizing the CRRA utility function with the coefficient of relative risk aversion equal to 3 and a subjective discount factor equal to 0.98, the subjective discount factor in this paper is $1/(1 + \rho)$, leads to a risk-free rate predicted by the model that is too high and an equity premium predicted by the model that is too low. If one were to use such a parameterization to calculate the welfare gains, one would discount the future by more than we should. Thus to match the risk-free interest rate and the equity premium to those found in the data, one can raise the coefficient of relative risk aversion to obtain an appropriate equity premium and set the subjective discount factor to match the risk-free interest rate [see Narayana R. Kocherlakota (1996) for an example].[15] Doing this would have a positive effect on the welfare gain. Lewis (2000) has some exercises along these lines with Epstein and Zin (1989) preferences and it greatly increases her welfare gains numbers.[16] One reason for separating the coefficient of relative risk aversion from the elasticity of intertemporal substitution is to try to match the equity premium and the risk-free interest rate with a parameterization of the utility function that is believable. Others have tried to match these by placing different

[14] In her model, infinite horizon economy with growth, if the risk-free interest rate is less than the risk-adjusted growth rate, then expected utility is infinite. Thus she must have the risk-free interest rate greater than the risk-adjusted growth rate. Equating our risk-free interest rate to her risk-free rate less risk-adjusted growth rate guarantees a positive risk-free interest rate for our economy. If the risk-free interest rate in our model were negative, and greater than -1, then we would obtain negative infinite expected utility [see equation (A13)].

[15] See Kocherlakota (1990) for the existence of equilibria in growth economies when the subjective discount factor is greater than one. The subjective discount factor will need to be greater than one to match the risk-free interest rate.

[16] She matches the equity premium and the risk-free interest rate by appropriately setting the coefficient of relative risk aversion and the elasticity of intertemporal substitution.

types of frictions in their models as well as modeling the stochastic processes governing income in a judicious manner [see John Heaton and Deborah J. Lucas (1996) and George M. Constantines and Duffie (1996)]. There is still much research to be done in this area. These results suggest that the welfare gains properly measured may not be low at all.

In our parameterization, we assumed a zero expected change in incomes for all countries and serially uncorrelated income shocks, whereas in Lewis (2000), there was an estimated positive growth rate, the same for all countries, and persistence in the shocks to the log of endowments. Thus, she made quite different assumptions about the endowment process. Other studies show yet more of a variety of assumptions about endowment processes. Athanasoulis and van Wincoop (2000) compare five studies to their own method [Harold L. Cole and Maurice Obstfeld (1991); van Wincoop (1994); Linda L. Tesar (1995); van Wincoop (1999); and Lewis (2000)] by estimating the welfare gains from risk sharing when only the stochastic process for endowments differs across models, imposing a common utility function, CRRA, and parameterization for all of them. They concluded that varying the assumptions about the stochastic process for endowments has a large effect on welfare gains from risk sharing. In general the more persistent are the shocks, the larger are the welfare gains.[17] One additional reason why the welfare gains are large in this paper, in addition to the parameterization of the utility function, is that our shocks to the endowment process are persistent.

One question that may arise is whether the gains from risk sharing are as large as estimated because we use income rather than consumption as the endowment process. Consumption takes account of risk sharing that has already taken place in the economy. If we use consumption as the endowment process, it will let us know how much risk sharing is left. For sensitivity analysis, we reestimated the welfare gains for all six contracts using consumption from the Penn World Table and we obtained an unweighted average welfare gain across countries of 4.14 percent. When income is the endowment process the unweighted average is 4.93 percent, resulting in a 0.79 percent reduction in welfare gain. Thus the contracts are still potentially very valuable.

Another possible problem in interpreting our results is suggested by the investors' "home bias" that has been described by a number of authors, notably Tesar and Werner (1995), as well as Lewis (1996) and Marianne Baxter and Urban Jermann (1997). The fact that most people appear to be reluctant to invest much abroad in today's markets may reflect some misrepresentation of investor preferences in our model. This would tend to suggest that our contracts would not be as important as estimated. On the other hand, the home bias is by some accounts a reflection of current institutions and political uncertainties, and investors' current lack of knowledge about foreign investments. In the future, some of these obstacles to foreign investment are likely to diminish.

V. Summary and Conclusion

We have presented a constant absolute risk premium model of the world economy that takes as given preferences and income processes, and shows how risk premia and the world real interest rate are determined in equilibrium. We have shown a correspondence between the world real interest rate and social welfare, and we have shown which risk-management contracts should be created to achieve the risk-optimal interest rate; they are the WIC contracts.

The application, using the three-level income model, to per capita income data on countries and derivation of WIC contracts illustrates some very important risk-management contracts that might be considered to be traded on new markets. We believe that the derivation of the WIC contracts, although based on limited data, are suggestive enough

[17] The results of John Y. Campbell and N. Gregory Mankiw (1987) suggest that even the random-walk assumption of our model may cause our method to understate, rather than overstate, the welfare gains. In the Campbell–Mankiw estimated time-series model for per capita real national income not only was there a unit root, but also shocks to national income build for a while after being first experienced.

that we can consider creating something approximating the first two WIC contracts. In application it would be a good idea to simplify them, such as equalizing the weights that are similar and setting the small weights to zero. That is, we should consider a U.S. versus Japan and a U.S. versus Core-EU per capita income swap. Our analysis indicates that it would be better *not* to lump other countries into these first contracts, not to do a swap involving the entire EU on one side, for example. Further econometric work should be conducted to confirm or reject the conclusions from the simple econometric analysis here.

Our theoretical framework has the potential to provide the foundation for econometric work that will suggest other, and better, definitions of WICs. The variance matrix, estimated from the three-level income model, may be further refined and account may be taken of other factors in income risks besides country factors. Further work may also generalize the assumptions about the stochastic process of income, may move to an overlapping-generation framework, or may incorporate endogenous investments in physical capital.

Appendix: No-Ponzi-Game Condition—Discussion

The no-Ponzi-game condition simply states that the present value of one's wealth at time T as T approaches infinity must be nonnegative. If it were allowed to be negative, then one would always incur debts and never repay them. To write the no-Ponzi-game condition as in equation (8), we follow Magill and Quinzii (1994), who use the concept of competitive price perceptions of Sanford J. Grossman and Oliver D. Hart (1979). Competitive price perceptions mean that agents use their own inter-temporal marginal rate of substitution to evaluate future income streams. We can then show that the equivalent probability measure under which individual i takes expectations in equation (8), are consistent with first-order conditions.[18] We can further show that under this new probability measure, prices plus total accumulated dividends after a normalization follows a martingale, which is well known.[19]

Derivation of Equation (32)

To obtain equation (32) we begin by solving equation (26) backward to time zero to obtain

(A1)

$$B_{ti} = \sum_{\tau=0}^{t-1} (y_{t-\tau i} + \mathbf{q}_i \mathbf{X}'_{t-\tau} + \mathbf{q}_i \bar{\mathbf{D}}' - c_{t-\tau i}) \times (1+r)^{\tau} + (1+r)^t B_{0i}.$$

We can write the consumption and income processes as

$$\text{(A2)} \quad c_{t-\tau i} = \mu_i(t-\tau) + c_{0i} + \sum_{k=1}^{t-\tau} v_{ki},$$

$$\text{(A3)} \quad y_{t-\tau i} = y_{0i} + \sum_{k=1}^{t-\tau} \varepsilon_{ki},$$

and

$$\text{(A4)} \quad \mathbf{X}'_{t-\tau} = \sum_{k=1}^{t-\tau} \mathbf{A}' \boldsymbol{\varepsilon}'_k.$$

Substituting equations (A2)–(A4) and (29) and (31) into equation (A1) we obtain

[18] See Chi-fu Huang and Robert H. Litzenberger (1988) for an example of how to construct this measure.

[19] For the general theory of equivalent martingale measures and no arbitrage, see J. Michael Harrison and David M. Kreps (1979). Duffie (1996) also covers this, though in a different fashion. For a less terse handling of these issues see Huang and Litzenberger (1988). For a discussion of equivalent probability measures in an infinite horizon setting, see Huang and Henri Pagès (1992). An unpublished Appendix available from the authors provides a further discussion on the budget constraint and the no-Ponzi-game condition.

$$\text{(A5)} \quad B_{ti} = \sum_{\tau=0}^{t-1} \Bigg[y_{0i} + \mathbf{q}_i \bar{\mathbf{D}}' - \mu_i (t-\tau) - y_{0i} - \frac{1}{1+r} \mathbf{q}_i \bar{\mathbf{D}}' + \frac{1}{r} \mu_i - \frac{r}{1+r} B_{0i}^{-} \Bigg] (1+r)^{\tau} + (1+r)^t \left(-\frac{1}{1+r} \mathbf{q}_i \bar{\mathbf{D}}' + \frac{1}{r} \mu_i + \frac{1}{1+r} B_{0i}^{-} \right).$$

We can rewrite this expression as

$$\text{(A6)} \quad B_{ti} = -\frac{1}{1+r} \mathbf{q}_i \bar{\mathbf{D}}' + \mu_i (1+r)^t \left[\frac{1}{r} \sum_{\tau=0}^{t} (1+r)^{-\tau} - \sum_{\tau=0}^{t} \tau (1+r)^{-\tau} \right] + \frac{1}{1+r} B_{0i}^{-}.$$

Noting that $\mu_i(1+r)^t[(1/r)\sum_{\tau=0}^{t}(1+r)^{-\tau} - \sum_{\tau=0}^{t}\tau(1+r)^{-\tau}] = (1/r)\mu_i(t+1)$ gives us the result.

Derivation of Equation (39)

To obtain equation (39) we begin with expected lifetime utility at time zero as

$$\text{(A7)} \quad U_{0i} = \mathrm{E}_0 \left[\sum_{\tau=0}^{\infty} (-\exp(-\gamma c_{\tau i}))/(1+\rho)^{\tau} \right]$$

and the consumption process

$$\text{(A8)} \quad c_{ti} = \mu_i t + c_{0i} + \sum_{k=1}^{t} v_{ki}.$$

If we substitute (A8) into (A7) we obtain

$$\text{(A9)} \quad U_{0i} = -\exp(-\gamma c_{0i}) - \mathrm{E}_0 \left[\sum_{\tau=1}^{\infty} \left(\frac{1}{1+\rho} \right)^{\tau} \times \exp\left(-\gamma \left(\mu_i \tau + c_{0i} + \sum_{k=1}^{\tau} v_{ki} \right) \right) \right].$$

Noting that v_{ki} is normally distributed with mean zero and variance $\sigma_{v_i}^2$ for all k, one can rewrite equation (A9) as

$$\text{(A10)} \quad U_{0i} = -\exp(-\gamma c_{0i}) \left[\sum_{\tau=0}^{\infty} \left(\frac{1}{1+\rho} \right)^{\tau} \times \exp\left(-\gamma \mu_i \tau + \frac{1}{2} \gamma^2 \sigma_{v_i}^2 \tau \right) \right].$$

We can see from equation (18) that

$$\text{(A11)} \quad \gamma \mu_i = \ln(1+r) - \ln(1+\rho) + \tfrac{1}{2} \gamma^2 \sigma_{v_i}^2.$$

Note also that

$$\text{(A12)} \quad (1+\rho)^{-\tau} = \exp(-\tau \ln(1+\rho)).$$

Substituting equations (A12) and (A11) into (A10) we obtain

$$\text{(A13)} \quad U_{0i} = -\exp(-\gamma c_{0i}) \sum_{\tau=0}^{\infty} (1+r)^{-\tau},$$

and evaluating the infinite sum gives us the desired result.

Derivation of Equation (41)

First substitute (22) into (35). Because we are maximizing $\ln(1 + r)$ with respect to $\mathbf{A}$ and ρ is unaffected by the choice of $\mathbf{A}$, we can see that we will be minimizing the one-period average variance of consumption in the economy. Note now that we can rewrite the average variance of consumption in the economy as

$$\text{(A14)} \qquad \frac{1}{I}\left(\mathrm{tr}[\boldsymbol{\Sigma}] - \mathrm{tr}[\mathbf{A}'\boldsymbol{\Sigma}\mathbf{M}\boldsymbol{\Sigma}\mathbf{A}]\right).$$

Because $\boldsymbol{\Sigma}$ is not affected by the choice of the contract designer, the result follows.

Derivation of Equation (50)

To derive this equation we use equation (39) to obtain the following two equations:

$$\text{(A15)} \qquad U_{0i} = -\frac{1+r}{r}\exp(-\gamma c_{0i})$$

and

$$\text{(A16)} \qquad U_{0i}^{-} = -\frac{1+r_{-}}{r_{-}}\exp(-\gamma c_{0i}^{-})\exp(-\gamma F_{iN}).$$

When we equate these two expressions, as in equation (49), $U_{0i} = U_{0i}^{-}$, we obtain

$$\text{(A1)} \qquad \exp(-\gamma F_{iN}) = \exp(-\gamma(c_{0i} - c_{0i}^{-}))\frac{(1+r)r_{-}}{(1+r_{-})r}.$$

Taking the natural logarithm of both sides and dividing by γ gives the desired result.

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